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Magnetic effects in non-relativistic quantum electrodynamics: image corrections to the electron moment

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Abstract. Conventional non-relativistic quantum electrodynamics is used to calculate position-dependent corrections to the magnetic moment of an electron a distance z from a perfectly conducting surface. To leading order, $\delta\mu_{\parallel} = -e^3/32m^2z$ and $\delta\mu_z = 0$. These results follow easily and without any special pleading from the usual Foldy–Wouthuysen Hamiltonian, and are validated by necessarily much more elaborate relativistic calculations using either the full quantum field theory, or Dirac single-particle theory. Together with a recent non-relativistic account of the free-electron anomalous moment by Grotch and Kazes, such agreement is interpreted as a plausible indication that non-relativistic quantum electrodynamics is at least qualitatively adequate to deal with magnetic radiative effects, on the same footing as it can deal with non-magnetic analogues like the Lamb shift.

1. Introduction

It has long been the common view that explicitly relativistic considerations are essential to account, even qualitatively, for the anomalous magnetic moment $\delta\mu = e^3/4\pi m$ of the free electron. (See for instance Feynman (1961); for a contrary view, see Arunasalam (1969).) Ever since Welton's (1948) paper, simple non-relativistic approaches were believed to yield the wrong sign of $\delta\mu$. If this were true, it would force any correct heuristic explanation of $\delta\mu$ onto an altogether different footing from, say, the Bethe theory of the Lamb shift (see e.g. Power 1964); and it would largely sap one's confidence in non-relativistic quantum electrodynamics as an intuitive guide to magnetic effects. However, the common view has recently been disproved (Grotch and Kazes 1977), stemming as it did from failure to express the observed total magnetic moment $\mu = e/2m + \delta\mu$ in terms of the observed mass m. Once this is done, and if a physically reasonable cut-off is adopted, the correct sign and order of magnitude of $\delta\mu$ follow from an entirely straightforward calculation proceeding without any special tricks from the conventional Foldy–Wouthuysen (FW) Hamiltonian $H(\text{FW})$.

The present paper aims to provide further evidence for the (approximate) validity of the non-relativistic approach even to magnetic effects. To this end we consider an electron moving freely outside a perfectly conducting half-space occupying the region $z \leq 0$ and subject to an external (unquantized) constant homogeneous magnetic field $\tilde{\boldsymbol{B}}$. The boundary conditions at the surface of the conductor constrain the normal modes of

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the quantized Maxwell field; thereby they are responsible not only for the usual spin-independent 'dynamic image potential' (see e.g. Barton 1977) but also for a position-dependent correction to the electron magnetic moment. To leading order in e, and in the inverse distance z^{-1} of the electron from the surface, such magnetic corrections are proportional to $(e^3/m^2z)\sigma_z\tilde{B}_z$ and $(e^3/m^2z)\boldsymbol{\sigma}_\parallel\cdot\tilde{\boldsymbol{B}}_\parallel$, as can be seen from dimensional and invariance arguments (the suffixes z and $\parallel$ identify vector components normal and parallel, respectively, to the surface); they are obvious generalizations of the energy $-\delta\mu\boldsymbol{\sigma}\cdot\tilde{\boldsymbol{B}}$ for an isolated electron.

Although these effects are somewhat artificial, and unlikely to become observable in practice, they are observable in principle, and have the advantage, as compared to $\delta\mu$ itself, of being cut-off independent and mathematically well defined, even non-relativistically. In § 2 we calculate them from H(FW); the status of these results is established in § 3, which re-derives them to leading order from relativistic theory, as briefly as possible. Before the rehabilitation of the non-relativistic approach to magnetic effects, only this incomparably more laborious method could have been trusted even qualitatively; see Babiker and Barton (1972) for a closely related relativistic calculation performed under just this mistaken impression; (its result was incorrectly interpreted in that paper and is commented on in § 4 below). In § 4 we outline our conclusions.

2. Non-relativistic calculation

The standard non-relativistic Hamiltonian, correctly to order m^{-2}, is

$$H(\mathrm{FW}) = H_{\mathrm{rad}} + \frac{1}{2m}(\boldsymbol{p} - e\boldsymbol{A} - e\tilde{\boldsymbol{A}})^2 - \frac{e}{2m}\boldsymbol{\sigma}\cdot(\boldsymbol{B}+\tilde{\boldsymbol{B}}) - \frac{e^2}{4z}$$
$$-\frac{e}{4m^2}\boldsymbol{\sigma}\cdot\left(\boldsymbol{E} - \hat{\boldsymbol{z}}\frac{e}{4z^2}\right)\times(\boldsymbol{p} - e\boldsymbol{A} - e\tilde{\boldsymbol{A}}). \tag{2.1}$$

Here, the electron coordinate is $\boldsymbol{r} = (\boldsymbol{\rho}, z)$; $-e^2/4z$ is the electrostatic image potential, and $-e\hat{\boldsymbol{z}}/4z^2$ the corresponding electric field; the external (constant) magnetic field $\tilde{\boldsymbol{B}}$ derives from the vector potential

$$\tilde{\boldsymbol{A}}(\boldsymbol{r}) = \tfrac{1}{2}\tilde{\boldsymbol{B}}\times\boldsymbol{r}; \tag{2.2}$$

$\boldsymbol{A}$, $\boldsymbol{E} \equiv -\dot{\boldsymbol{A}}$, $\boldsymbol{B} \equiv \nabla\times\boldsymbol{A}$ are second-quantized field operators, with $\boldsymbol{A}$ given by (Barton 1974, see Barton 1977 for a discussion of the choice of gauge):

$$\boldsymbol{A} = \frac{1}{\pi}\int_0^\infty \mathrm{d}l \int_{-\infty}^{\infty} \mathrm{d}^2k \frac{1}{\omega^{1/2}}\bigg[a_1(\boldsymbol{k}, l)\hat{\boldsymbol{k}}\times\hat{\boldsymbol{z}}\sin(lz)$$
$$+ a_2(\boldsymbol{k}, l)\left(\hat{\boldsymbol{k}}\frac{il}{\omega}\sin(lz) - \hat{\boldsymbol{z}}\frac{k}{\omega}\cos(lz)\right)\bigg]\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{\rho}} + \mathrm{HC}. \tag{2.3}$$

In (2.3), $\omega = (\boldsymbol{k}^2 + l^2)^{1/2}$; the a's are annihilation operators, i.e. $[a_s(\boldsymbol{k}, l), a_{s'}^\dagger(\boldsymbol{k}', l')] = \delta_{ss'}\delta^{(2)}(\boldsymbol{k} - \boldsymbol{k}')\delta(l - l')$ and HC stands for Hermitean conjugate. Finally, in (2.1), H_{rad} is the Hamiltonian for the radiation field in absence of the electron:

$$H_{\mathrm{rad}} = \sum_{s=1,2}\int_0^\infty \mathrm{d}l\int_{-\infty}^{\infty}\mathrm{d}^2k\,\omega a_s^\dagger(\boldsymbol{k}, l)a_s(\boldsymbol{k}, l). \tag{2.4}$$

The only unconventional feature of H(FW) is the absence of a Darwin term; this is discussed briefly at the end of § 3, though it is irrelevant to the effects we are pursuing.

We consider an electron described by a quasiclassically moving wavepacket which does not impinge on the conductor during the experiment; as explained in the introduction we require those terms in the energy which are: (i) spin-dependent; (ii) linear in $\tilde{\boldsymbol{B}}$; (iii) of order e^3; and (iv) of order m^{-2}. Inspection of (2.1) and a little reflexion show that there are only two such contributions; first, the spin–orbit (SL) coupling from the last term in (2.1),

$$\Delta E_{\mathrm{SL}}(\mathrm{FW}) = -\frac{e^3}{16m^2z^2}\boldsymbol{\sigma}\cdot\hat{\boldsymbol{z}}\times\tilde{\boldsymbol{A}} = -\frac{e^3}{32m^2z}\boldsymbol{\sigma}_{\|}\cdot\tilde{\boldsymbol{B}}_{\|}, \tag{2.5}$$

and second, the second-order perturbation stemming from interference between the couplings $-e\boldsymbol{\sigma}\cdot\boldsymbol{B}/2m$ and $e^2\boldsymbol{A}\cdot\tilde{\boldsymbol{A}}/m$ of the electron to the quantized radiation field. Writing $\boldsymbol{A}$ and $\boldsymbol{B}$ symbolically as $\Sigma_\lambda(a_\lambda\boldsymbol{A}_\lambda + a_\lambda^\dagger\boldsymbol{A}_\lambda^*)$ and $\Sigma_\lambda(a_\lambda\boldsymbol{B}_\lambda + a_\lambda^\dagger\boldsymbol{B}_\lambda^*)$, with $\boldsymbol{B}_\lambda = \nabla\times\boldsymbol{A}_\lambda$, we require the spin- and z-dependent parts of

$$-\sum_\lambda\frac{1}{\omega_\lambda}\left|-\frac{e}{2m}\boldsymbol{\sigma}\cdot\boldsymbol{B}_\lambda + \frac{e^2}{m}\boldsymbol{A}_\lambda\cdot\tilde{\boldsymbol{A}}\right|^2.$$

Using the explicit expressions for $\boldsymbol{A}_\lambda$ from (2.3), and for $\boldsymbol{B}_\lambda$, and writing $\int_0^\infty \mathrm{d}k\, k\ldots = \int_l^\infty \mathrm{d}\omega\,\omega\ldots$, we find after some straightforward manipulation

$$\Delta E_{\mathrm{int}}(\mathrm{FW}) = \frac{e^3 z}{4m^2}\boldsymbol{\sigma}_{\|}\cdot\tilde{\boldsymbol{B}}_{\|}\frac{2}{\pi}\int_0^\infty \mathrm{d}l\, l\sin(2lz)\int_l^\infty\frac{\mathrm{d}\omega}{\omega}. \tag{2.6}$$

The divergent upper limit of the ω integral enters only as a coefficient of $\delta'(2z)$, and we drop it because it is a contact interaction ineffective while the electron remains outside the conductor. For the lower-limit contribution $-\ln l$ we use

$$-\int_0^\infty \mathrm{d}l\, l\sin(2lz)\ln l = \frac{1}{2}\frac{\partial}{\partial z}\int_0^\infty \mathrm{d}l\cos(2lz)\ln l = \frac{1}{2}\frac{\partial}{\partial z}\left(-\frac{\pi}{4z}\right) = \frac{\pi}{8z^2}; \tag{2.7}$$

this and other similar technical devices are discussed elsewhere (Barton 1974, 1977). Thus one finds

$$\Delta E_{\mathrm{int}}(\mathrm{FW}) = \frac{e^3}{16m^2z}\boldsymbol{\sigma}_{\|}\cdot\tilde{\boldsymbol{B}}_{\|}, \tag{2.8}$$

which combines with (2.5) to give the end result

$$\Delta E(\mathrm{FW}) = \frac{e^3}{32m^2z}\boldsymbol{\sigma}_{\|}\cdot\tilde{\boldsymbol{B}}_{\|}. \tag{2.9}$$

In other words the position-dependent changes in $\delta\mu$ are, to leading order,

$$\delta\mu_z = 0, \qquad \delta\mu_{\|} = -e^3/32m^2z. \tag{2.10}$$

3. Relativistic calculation

As motivated in the introduction, this section re-derives (2.9) as briefly as possible from the correct relativistic field theory (QED). We also show, at minimal extra cost in computation, that the same expression results also from relativistic single-particle theory (SPT) (see Grotch and Kazes 1976), and shall comment on this coincidence in § 4. At the end of the present section we discuss briefly the provenance of $H(\mathrm{FW})$, equation (2.1), which underlies the non-relativistic calculation.

The complete Hamiltonian is written as

$$H = H_0 + H_{\mathrm{es}} + H_{\mathrm{int}}. \tag{3.1}$$

H_0 contains H_{rad} (equation (2.4)) and the Hamiltonian for the electron in presence of the magnetic field $\tilde{\boldsymbol{B}}$. Thus

$$H_0(\mathrm{QED}) = H_{\mathrm{rad}} + \int \mathrm{d}\boldsymbol{r}\, \psi^\dagger(\beta m + \boldsymbol{\alpha} \cdot \boldsymbol{\pi})\psi, \tag{3.2}$$

$$H_0(\mathrm{SPT}) = H_{\mathrm{rad}} + \beta m + \boldsymbol{\alpha} \cdot \boldsymbol{\pi}, \tag{3.3}$$

where $\boldsymbol{\pi} = (\boldsymbol{p} - e\tilde{\boldsymbol{A}})$, and ψ is the usual second-quantized electron field operator.

In QED the electrostatic part is

$$H_{\mathrm{es}}(\mathrm{QED}) = \frac{1}{2} e^2 \iint \mathrm{d}\boldsymbol{r}\, \mathrm{d}\boldsymbol{r}'\, \psi^\dagger(\boldsymbol{r})\psi(\boldsymbol{r}) \left(\frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} - \frac{1}{|\boldsymbol{s} - \boldsymbol{r}'|} \right) \psi^\dagger(\boldsymbol{r}')\psi(\boldsymbol{r}'), \tag{3.4}$$

where $\boldsymbol{s} \equiv (x, y, -z)$. In SPT we can drop the Coulomb self-energy for an isolated electron (which is independent of position and momentum) and write

$$H_{\mathrm{es}}(\mathrm{SPT}) = -e^2/4z. \tag{3.5}$$

Finally, the couplings H_{int} to the quantized radiation field are

$$H_{\mathrm{int}}(\mathrm{QED}) = -e \int \mathrm{d}\boldsymbol{r}\, \psi^\dagger \boldsymbol{\alpha} \cdot \boldsymbol{A}\psi, \tag{3.6}$$

$$H_{\mathrm{int}}(\mathrm{SPT}) = -e\boldsymbol{\alpha} \cdot \boldsymbol{A}. \tag{3.7}$$

Mass- and wave-renormalization terms are position independent and have been dropped because they play no role in the order $-e^2 z$-dependent expressions of interest here.

To calculate the static image potential we start with a one-electron packet state, as discussed in § 2. Once we have extracted operators linear in $\tilde{\boldsymbol{B}}$ the state can be assumed to be at rest and therefore for purposes of carrying out perturbation theory we need only the unperturbed eigenstate $|n\rangle$, with

$$(\boldsymbol{\alpha} \cdot \boldsymbol{\pi} + \beta m)|n\rangle = E_n |n\rangle. \tag{3.8}$$

We have found it convenient to set up the calculation in a way which parallels the anomalous-moment calculation of Grotch and Kazes (1977), separating the energy shift into an electrostatic part ΔE_{es} and a part ΔE_{int} due to the quantized radiation field.

The z-dependent parts of the shifts ΔE_{es} may be shown to be given by

$$\Delta E_{\mathrm{es}}(\mathrm{SPT}) = -\frac{e^2}{4\pi^2} \int \mathrm{d}l\, \mathrm{d}^2k\, \frac{1}{\omega^2} \langle n| \mathrm{e}^{-2ilz} (\Lambda_+(\boldsymbol{\pi} - \boldsymbol{k} - \hat{\boldsymbol{z}}l) + \Lambda_-(\boldsymbol{\pi} - \boldsymbol{k} - \hat{\boldsymbol{z}}l))|n\rangle \tag{3.9}$$

and

$$\Delta E_{\mathrm{es}}(\mathrm{QED}) = -\frac{e^2}{4\pi^2} \int \mathrm{d}l\, \mathrm{d}^2k\, \frac{1}{\omega^2} \langle n| \mathrm{e}^{-2ilz} (\Lambda_+(\boldsymbol{\pi} - \boldsymbol{k} - \hat{\boldsymbol{z}}l) - \Lambda_-(\boldsymbol{\pi} - \boldsymbol{k} - \hat{\boldsymbol{z}}l))|n\rangle, \tag{3.10}$$

where the projection operators are

$$\Lambda_{\pm}(\boldsymbol{\pi}-\boldsymbol{k}-\hat{\boldsymbol{z}}l)=\frac{E(\boldsymbol{\pi}-\boldsymbol{k}-\hat{\boldsymbol{z}}l)\pm[\boldsymbol{\alpha}\cdot(\boldsymbol{\pi}-\boldsymbol{k}-\hat{\boldsymbol{z}}l)+\beta m]}{2E(\boldsymbol{\pi}-\boldsymbol{k}-\hat{\boldsymbol{z}}l)}, \tag{3.11}$$

and $E(\boldsymbol{\pi}-\boldsymbol{k}-\hat{\boldsymbol{z}}l)=+\{[\boldsymbol{\alpha}\cdot(\boldsymbol{\pi}-\boldsymbol{k}-\hat{\boldsymbol{z}}l)+\beta m]^2\}^{1/2}$. In the above expressions all three integrals run from $-\infty$ to $+\infty$. These shifts simplify to

$$\Delta E_{\rm es}(\text{SPT})=-\frac{e^2}{4\pi^2}\int \mathrm{d}l\,\mathrm{d}^2k\,\frac{1}{\omega^2}\langle n|\,\mathrm{e}^{-2\mathrm{i}lz}|n\rangle \tag{3.12}$$

and

$$\Delta E_{\rm es}(\text{QED})=-\frac{e^2}{4\pi^2}\int \mathrm{d}l\,\mathrm{d}^2k\,\frac{1}{\omega^2}\left\langle n\left|\,\mathrm{e}^{-2\mathrm{i}lz}\left(\frac{\boldsymbol{\alpha}\cdot(\boldsymbol{\pi}-\boldsymbol{k}-\hat{\boldsymbol{z}}l)+\beta m}{E(\boldsymbol{\pi}-\boldsymbol{k}-\hat{\boldsymbol{z}}l)}\right)\right|n\right\rangle. \tag{3.13}$$

Let us now turn to the expressions for $\Delta E_{\rm int}$. The difference between SPT and QED appears in the structure of the electron propagator (Grotch and Kazes 1976). For brevity we write $\Delta E_{\rm int}(+)\equiv\Delta E_{\rm int}(\text{QED})$ and $\Delta E_{\rm int}(-)\equiv\Delta E_{\rm int}(\text{SPT})$. Then by standard second-order perturbation-theory techniques we can show that the energy shifts are given by

$$\begin{aligned}\Delta E_{\rm int}(\pm)=\frac{e^2}{2\pi^2}\int_{-\infty}^{\infty}\mathrm{d}l\int_{-\infty}^{\infty}\mathrm{d}^2k\frac{1}{\omega}\Bigg[&\left\langle n\left|\boldsymbol{\alpha}\cdot\hat{\boldsymbol{k}}\times\hat{\boldsymbol{z}}\sin(lz)\right.\right.\\ &\times\left(\frac{\Lambda_+(\boldsymbol{\pi}-\boldsymbol{k})}{E_n-\omega-E(\boldsymbol{\pi}-\boldsymbol{k})}+\frac{\Lambda_-(\boldsymbol{\pi}-\boldsymbol{k})}{E_n\pm\omega+E(\boldsymbol{\pi}-\boldsymbol{k})}\right)\boldsymbol{\alpha}\cdot\hat{\boldsymbol{k}}\times\hat{\boldsymbol{z}}\sin(lz)\left.\left|n\right.\right\rangle\\ &+\left\langle n\left|\boldsymbol{\alpha}\cdot\left(\hat{\boldsymbol{k}}\frac{\mathrm{i}l}{\omega}\sin(lz)-\hat{\boldsymbol{z}}\frac{k}{\omega}\cos(lz)\right)\right.\right.\\ &\times\left(\frac{\Lambda_+(\boldsymbol{\pi}-\boldsymbol{k})}{E_n-\omega-E(\boldsymbol{\pi}-\boldsymbol{k})}+\frac{\Lambda_-(\boldsymbol{\pi}-\boldsymbol{k})}{E_n\pm\omega+E(\boldsymbol{\pi}-\boldsymbol{k})}\right)\\ &\times\boldsymbol{\alpha}\cdot\left(-\hat{\boldsymbol{k}}\frac{\mathrm{i}l}{\omega}\sin(lz)-\hat{\boldsymbol{z}}\frac{k}{\omega}\cos(lz)\right)\left.\left|n\right.\right\rangle\Bigg].\end{aligned} \tag{3.14}$$

The sines and cosines are rewritten in terms of complex exponentials which act as momentum shift operators. We obtain for the z-dependent part of the energy shift

$$\begin{aligned}\Delta E_{\rm int}(\pm)=-\frac{e^2}{4\pi^2}\int \mathrm{d}l\,\mathrm{d}^2k\frac{1}{\omega}\big(&\langle n|\boldsymbol{\alpha}\cdot\hat{\boldsymbol{u}}_1\,\mathrm{e}^{-2\mathrm{i}lz}S_{\pm}(\boldsymbol{\pi}-\boldsymbol{k}-\hat{\boldsymbol{z}}l)\boldsymbol{\alpha}\cdot\hat{\boldsymbol{u}}_1|n\rangle\\ &+\langle n|\boldsymbol{\alpha}\cdot\hat{\boldsymbol{u}}_3\,\mathrm{e}^{-2\mathrm{i}lz}S_{\pm}(\boldsymbol{\pi}-\boldsymbol{k}-\hat{\boldsymbol{z}}l)\boldsymbol{\alpha}\cdot\hat{\boldsymbol{u}}_2|n\rangle\big),\end{aligned} \tag{3.15}$$

with $\hat{\boldsymbol{u}}_1=\hat{\boldsymbol{k}}\times\hat{\boldsymbol{z}}$, $\hat{\boldsymbol{u}}_2=(\hat{\boldsymbol{k}}l-\hat{\boldsymbol{z}}k)/\omega$, $\hat{\boldsymbol{u}}_3=(\hat{\boldsymbol{k}}l+\hat{\boldsymbol{z}}k)/\omega$, and

$$S_{\pm}(\boldsymbol{\pi}-\boldsymbol{k}-\hat{\boldsymbol{z}}l)=\frac{\Lambda_+(\boldsymbol{\pi}-\boldsymbol{k}-\hat{\boldsymbol{z}}l)}{E_n-\omega-E(\boldsymbol{\pi}-\boldsymbol{k}-\hat{\boldsymbol{z}}l)}+\frac{\Lambda_-(\boldsymbol{\pi}-\boldsymbol{k}-\hat{\boldsymbol{z}}l)}{E_n\pm\omega+E(\boldsymbol{\pi}-\boldsymbol{k}-\hat{\boldsymbol{z}}l)}. \tag{3.16}$$

Equations (3.12), (3.13) and (3.15) constitute the starting point for the relativistic SPT or QED calculation of z-dependent corrections to the magnetic moment, as well as for z-dependent corrections in the absence of a $\tilde{\boldsymbol{B}}$ field (which we do not derive here).

To sufficient accuracy the energy eigenvalue E_n used in the calculation is given by

$$E_n = m - \frac{e}{2m}\langle \boldsymbol{\sigma} \cdot \tilde{\boldsymbol{B}}\rangle + \frac{1}{2m}\langle \boldsymbol{\pi}^2\rangle, \tag{3.17}$$

but when the end results are applied to quasiclassical wave packets any effects due to $\langle \boldsymbol{\pi}^2\rangle/2m$ contribute only when the electron is in motion. Hereafter these motional effects will be dropped from E_n since we are interested only in the static image potential.

We discuss briefly the calculation of $\Delta E_{\mathrm{es}}(\mathrm{QED})$ to illustrate the procedure used. To order $1/m^2$ we may replace the term in large parentheses in equation (3.13) by

$$\frac{\boldsymbol{\alpha} \cdot (\boldsymbol{\pi} - \boldsymbol{k} - \hat{\boldsymbol{z}}l)}{m} + \beta\left(1 - \frac{(\boldsymbol{\pi} - \boldsymbol{k} - \hat{\boldsymbol{z}}l)^2}{2m^2} + \frac{e}{2m^2}\boldsymbol{\sigma} \cdot \tilde{\boldsymbol{B}}\right). \tag{3.18}$$

In making this expansion we realize that $\boldsymbol{\alpha}$ is of the order $1/m$ since in equation (3.13) it couples the upper and lower components of the Dirac state vector. Our procedure is now to reduce the Dirac matrix elements to Schrödinger matrix elements. To carry this out we express the Dirac state vector $|n\rangle$ in terms of the Schrödinger state vector $|n_0\rangle$ using the relation

$$|n\rangle = N\left(1 + \frac{\boldsymbol{\alpha} \cdot \boldsymbol{\pi}}{2m}\right)|n_0\rangle, \tag{3.19}$$

and the normalizations $\langle n|n\rangle = 1 = \langle n_0|n_0\rangle$. The normalization constant N has a momentum-dependent piece, which we drop, as well as a $\tilde{\boldsymbol{B}}$-dependent part. Thus $N^2 \approx 1 + e\langle \boldsymbol{\sigma} \cdot \tilde{\boldsymbol{B}}\rangle/4m^2$.

Using equations (3.13), (3.18) and (3.19) we can extract the static magnetic-field dependent energy shift of order $1/m^2$. The calculation is straightforward but considerable care is needed in maintaining the order of operators and in extracting the static contribution†. We find that the z-dependent magnetic-field correction from equation (3.13) is

$$\begin{aligned}\Delta E_{\mathrm{es}}(\mathrm{QED}) &= -\frac{e^2}{4\pi^2}\int \mathrm{d}l\, \mathrm{d}^2k\, \frac{1}{\omega^2}\left\langle n_0\right| \mathrm{e}^{-2ilz}\frac{ielz}{4m^2}\boldsymbol{\sigma}_{\parallel} \cdot \tilde{\boldsymbol{B}}_{\parallel}\left|n_0\right\rangle \\ &= \left\langle n_0\right| -\frac{e^3}{32m^2 z}\boldsymbol{\sigma}_{\parallel} \cdot \tilde{\boldsymbol{B}}_{\parallel}\left|n_0\right\rangle.\end{aligned} \tag{3.20}$$

Thus we have a magnetic-field-dependent potential of $-e^3\boldsymbol{\sigma}_{\parallel} \cdot \tilde{\boldsymbol{B}}_{\parallel}/32m^2 z$ arising from the electrostatic interaction in field theory. The SPT result has also been evaluated to order $1/m^2$ and gives exactly the same result:

$$\Delta E_{\mathrm{es}}(\mathrm{SPT}) = \left\langle n_0\right| -\frac{e^3}{32m^2 z}\boldsymbol{\sigma}_{\parallel} \cdot \tilde{\boldsymbol{B}}_{\parallel}\left|n_0\right\rangle. \tag{3.21}$$

† As an example of the sort of difficulty which can arise consider the matrix element of $\mathrm{e}^{-2ilz}\boldsymbol{\sigma} \cdot \boldsymbol{p}$ in a packet state. To extract the momentum-independent or static portion we rewrite the operator as $\frac{1}{2}\{\mathrm{e}^{-2ilz}, \boldsymbol{\sigma} \cdot \boldsymbol{p}\} + \frac{1}{2}[\mathrm{e}^{-2ilz}, \boldsymbol{\sigma} \cdot \boldsymbol{p}]$. The anticommutator can be shown to give a contribution which is proportional to the particle velocity whereas the commutator gives a static contribution. If the same operator was evaluated in a momentum eigenstate by applying the operator $\boldsymbol{p}$ directly to the state, the wrong answer would emerge since the static contribution would be missing.

Next we turn our attention to the contributions arising from equation (3.15). The propagation functions may be considerably simplified since again we are only interested in energy shifts of order $1/m^2$. To this accuracy we can write

$$S_\pm(\boldsymbol{\pi}-\boldsymbol{k}-\hat{\boldsymbol{z}}l) \approx -\frac{1+\beta}{2\omega}-\frac{\boldsymbol{\alpha}\cdot(\boldsymbol{\pi}-\boldsymbol{k}-\hat{\boldsymbol{z}}l)}{2m\omega}+\frac{1}{2m}-\frac{1}{4m^2\omega}\left(\pm\omega^2+(\boldsymbol{\pi}-\boldsymbol{k}-\hat{\boldsymbol{z}}l)^2-e\boldsymbol{\sigma}\cdot\tilde{\boldsymbol{B}}\right). \tag{3.22}$$

We note that the difference between SPT and QED resides entirely in the difference between S_+ and S_-. Since this difference is independent of $\tilde{\boldsymbol{B}}$, and already of order $1/m^2$, we expect to obtain the same result in both theories.

The calculation using equations (3.15), (3.19) and (3.22) is straightforward but tedious. Again some care is necessary for extracting the static terms. We find

$$\Delta E_{\text{int}}(\text{QED}) = \Delta E_{\text{int}}(\text{SPT}) = \left\langle n_0 \left| \frac{e^3}{16m^2z}\,\boldsymbol{\sigma}_\parallel\cdot\tilde{\boldsymbol{B}}_\parallel \right| n_0 \right\rangle. \tag{3.23}$$

Combining this result with the electrostatic contribution we obtain the total magnetic potential

$$\Delta E = \frac{e^3}{32m^2z}\,\boldsymbol{\sigma}_\parallel\cdot\tilde{\boldsymbol{B}}_\parallel. \tag{3.24}$$

This leads to exactly the same magnetic-moment correction as that obtained in equation (2.9) using the much simpler FW reduction.

We end this section with a brief comment on the derivation of $H(\text{FW})$ (equation (2.1)) from the relativistic SPT operator used above:

$$H(\text{SPT}) = H_{\text{rad}}+\beta m+\boldsymbol{\alpha}\cdot(\boldsymbol{p}-e\boldsymbol{A}-e\tilde{\boldsymbol{A}})-e^2/4z. \tag{3.25}$$

To $H(\text{SPT})$ one applies the standard sequence of canonical transformations $U = \ldots U_2U_1$ (Foldy and Wouthuysen 1950) beginning with $U_1 = \exp[\beta\boldsymbol{\alpha}\cdot(\boldsymbol{p}-e\boldsymbol{A}-e\tilde{\boldsymbol{A}})/2m]$. To our order, one obtains

$$UH(\text{SPT})U^{-1} = m+H(\text{FW})-\frac{1}{8m^2}\nabla^2\left(\frac{e^2}{4z}\right)-\frac{ie^2}{8m^2}[\boldsymbol{A},\dot{\boldsymbol{A}}]+O\left(\frac{1}{m^3}\right). \tag{3.26}$$

The third term is the familar Darwin interaction stemming from the image potential $-e^2/4z$ in (3.25). The feature to watch is that U fails to commute with H_{rad}; this is responsible not only for the familiar appearance of $\boldsymbol{E} \equiv -\dot{\boldsymbol{A}}$ in $H(\text{FW})$, but also for the less familiar fourth term in (3.26). For an isolated electron this is a position- and momentum-independent constant which can be dropped because, like the rest-mass energy m, it plays no further role in non-relativistic contexts. But in our case the term has a z-dependent component which is easily determined once we write it, in the symbolic notation of § 2, as

$$\frac{e^2}{4m^2}\sum_\lambda \omega_\lambda|\boldsymbol{A}_\lambda|^2.$$

On evaluation it is found precisely to cancel the Darwin term. This completes the justification of the Hamiltonian $H(\text{FW})$ used in § 2.

4. Comments and conclusions

Our central result is that the magnetic moment of a free electron near a perfectly conducting surface suffers the anisotropic and position-dependent radiative corrections (2.10), which are correct to leading order in e^2 (i.e. to overall order e^3) and to leading order in $1/mz$ (i.e. to overall order $1/m^2z$). We have no simple explanation why $\delta\mu_z$ remains zero to this order. A relativistic (and difficult) calculation of such magnetic effects for an electron between two conductors was reported earlier (Babiker and Barton 1972). The results in that paper, though correct in the Dirac (four-component) representation, were not reduced as they should have been to the Schrödinger (two-component) representation. When this is done (along the lines of § 3 above), the leading magnetic-moment correction changes from isotropy (proportionality to $\boldsymbol{\sigma}\cdot\tilde{\boldsymbol{B}}$) to proportionality to $\boldsymbol{\sigma}_{\parallel}\cdot\tilde{\boldsymbol{B}}_{\parallel}$, essentially because the image potential $-e^2/4z$ in the Dirac representation spawns further spin-dependent terms on reduction to the Schrödinger representation. After such reduction, these earlier results agree with those reported here in the appropriate limit when one conductor recedes to infinity.

The precise agreement in leading order between relativistic QED and SPT is unexpected. It will certainly not persist to higher orders in e^2, because QED never needs cut-offs while SPT does; nor is it a general feature of all such magnetic calculations to order e^2, as witness the expressions for the usual magnetic moment of an isolated electron (Grotch and Kazes 1977). We have not performed the calculations needed to establish whether, to leading order in e^2, the agreement persists to higher orders in $(mz)^{-1}$. On the other hand, the agreement between SPT and the non-relativistic result could have been foreseen, because H(FW) is unitary-equivalent to H(SPT).

Accordingly, what we deem most noteworthy is not that the non-relativistic approach gives the leading term exactly, but that, without invoking any *ad hoc* arguments, it gives an approximation to the overall result which is qualitatively adequate as to sign, order of magnitude, and type of anisotropy. Comparison between the formalisms of §§ 2 and 3 makes our second point, that the non-relativistic approach is incomparably simpler. The adequacy of the position-dependent radiative corrections calculated here, and of those in free space (Grotch and Kazes 1977), makes it plausible that non-relativistic quantum electrodynamics is competent to deal with radiative corrections to magnetic effects at least to a rough first approximation, in the same sense, and in the same way, in which it can deal with analogous non-magnetic effects like the Lamb shift. Its precise quantitative limits of validity remain to be explored, but are of less practical interest, because for demonstrably accurate expressions one must in any case fall back on the full relativistic field theory.

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